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Advanced Linear Algebra (MA 409)
Problem Sheet - 21

Invariant Subspaces and the Cayley Hamilton Theorem

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1. Label the following statements as true or false.

   (a) There exists a linear operator T with no T-invariant subspace.

   (b) If T is a linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.

   (c) Let T be a linear operator on a finite-dimensional vector space V, and let v and w be in V. If W is the T-cyclic subspace generated by v, W' is the T-cyclic subspace generated by w, and $W = W'$, then $v = w$.

   (d) If T is a linear operator on a finite-dimensional vector space V, then for any $v \in V$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

   (e) Let T be a linear operator on an n-dimensional vector space. Then there exists a polynomial $g(t)$ of degree n such that $g(T) = T_0$.

   (f) Any polynomial of degree n with leading coefficient $(-1)^n$ is the characteristic polynomial of some linear operator.

   (g) If T is a linear operator on a finite-dimensional vector space V, and if V is the direct sum of k T-invariant subspaces, then there is an ordered basis β for V such that $[T]_\beta$ is a direct sum of k matrices.

2. For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

   (a) $V = P_3(\mathbb{R})$, $T(f(x)) = f'(x)$, and $W = P_2(\mathbb{R})$

   (b) $V = P(\mathbb{R})$, $T(f(x)) = xf(x)$, and $W = P_2(\mathbb{R})$

   (c) $V = \mathbb{R}^3$, $T(a,b,c) = (a+b+c, a+b+c, a+b+c)$, and $W = \{(t,t,t) : t \in \mathbb{R}\}$

   (d) $V = C([0,1])$, $T(f(t)) = \left[\int_0^1 f(x)\,dx\right]t$, and $W = \{f \in V : f(t) = at + b \text{ for some } a \text{ and } b\}$

   (e) $V = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $W = \{A \in V : A^t = A\}$

3. Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

   (a) $\{0\}$ and V

   (b) $N(T)$ and $R(T)$

   (c) E_λ, for any eigenvalue λ of T

4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(T)$-invariant for any polynomial $g(t)$.

5. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

6. For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace W generated by the vector z.

   (a) $V = \mathbb{R}^4$, $T(a,b,c,d) = (a+b, b-c, a+c, a+d)$, and $z = e_1$.

   (b) $V = P_3(\mathbb{R})$, $T(f(x)) = f''(x)$, and $z = x^3$.

   (c) $V = M_{2\times 2}(\mathbb{R})$, $T(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   (d) $V = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

7. Prove that the restriction of a linear operator T to a T-invariant subspace is a linear operator on that subspace.

8. Let T be a linear operator on a vector space with a T-invariant subspace W. Prove that if v is an eigenvector of T_W with corresponding eigenvalue λ, then the same is true for T.

9. For each linear operator T and cyclic subspace W in Exercise 6, compute the characteristic polynomial of T_W in two ways, as in Example 6.

10. For each linear operator in Exercise 6, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of T_W (computed in Exercise 9) divides $f(t)$.

11. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

    (a) W is T-invariant.

    (b) Any T-invariant subspace of V containing v also contains W.

12. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

13. Prove that the polynomial $g(t)$ of Exercise 12 can always be chosen so that its degree is less than $\dim(W)$.

14. Use the Cayley-Hamilton theorem to prove "Cayley-Hamilton Theorem for matrices": Let A be a $n \times n$ matrix, and let $f(t)$ bt the characteristic polynomial of A. Then $f(A) = O$, the $n \times n$ zero matrix.

    Warning : If $f(t) = \det(A - tI)$ is the characteristic polynomial of A, it is tempting to "prove" that $f(A) = O$ by saying "$f(A) = \det(A - AI) = \det(O) = 0$." But this argument is nonsense. Why?

15. Let T be a linear operator on a finite-dimensional vector space V.

    (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

    (b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

16. Let A be an $n \times n$ matrix. Prove that

$$\dim(span(\{I_n, A, A^2, \ldots\})) \leq n.$$

17. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

   (a) Prove that A is invertible if and only if $a_0 \neq 0$.
   (b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

   (c) Use (b) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

18. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Hint: Use mathematical induction on k, expanding the determinant along the first row.

19. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $UT = TU$ if and only if $U = g(T)$ for some polynomial $g(t)$.

Hint: Suppose that V is generated by v. Choose $g(t)$ according to Exercise 12 so that $g(T)(v) = U(v)$.

20. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $T = cI$ for some scalar c.

21. Let T be a linear operator on a two-dimensional vector space V and suppose that $T \neq cI$ for any scalar c. Show that if U is any linear operator on V such that $UT = TU$, then $U = g(T)$ for some polynomial $g(t)$.

22. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in W$ for all i.

Hint: Use mathematical induction on k.

23. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable.

Hint: Use the result of Exercise 22.

24. (a) Recall that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $TU = UT$). Prove the converse of the above statment that if T and U are diagonalizable linear operators on a finite-dimensional vector space V such that $UT = TU$, then T and U are simultaneously diagonalizable.

    Hint: For any eigenvalue λ of T, show that E_λ is U-invariant, and apply Exercise 23 to obtain a basis for E_λ of eigenvectors of U.

    (b) Recall that that if A and B are simultaneously diagonalizable matrices, then A and B commute. State and prove a matrix version of (a).

25. Let T be a linear operator on an n-dimensional vector space V such that T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself.

    Hint: Use Exercise 22 to find a vector v such that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is linearly independent.

For the purposes of Exercises 26 through 31, T is a fixed linear operator on a finite-dimensional vector space V, and W is a nonzero T-invariant subspace of V. We require the following definition.

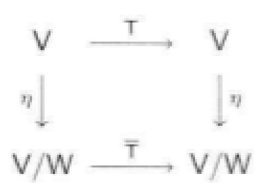
Definition. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Define $\overline{T} : V/W \to V/W$ by

$$\overline{T}(v + W) = T(v) + W \quad \text{for any} \quad v + W \in V/W.$$

26. (a) Prove that $\overline{T}$ is well defined. That is, show that $\overline{T}(v + W) = \overline{T}(v' + W)$ whenever $v + W = v' + W$.

    (b) Prove that $\overline{T}$ is a linear operator on V/W.

    (c) Let $\eta : V \to V/W$ be the linear transformation defined by $\eta(v) = v + W$. Show that the diagram of the following Figure commutes ; that is, prove that $\eta T = \overline{T}\eta$. (This exercise does not require the assumption that V is finite-dimensional.)



27. Let $f(t)$,$g(t)$, and $h(t)$ be the characteristic polynomials of T, T_W, and $\overline{T}$, respectively. Prove that $f(t) = g(t)h(t)$.

    Hint: Extend an ordered basis $\gamma = \{v_1, v_2, \ldots, v_k\}$ for W to an ordered basis

$$\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

    for V. Then show that the collection of cosets $\alpha = \{v_{k+1} + W, v_{k+2} + W, \ldots, v_n + W\}$ is an ordered basis for V/W, and prove that

$$[T]_\beta = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

    where $B_1 = [T]_\gamma$ and $B_3 = [\overline{T}]_\alpha$

28. Use the hint in Exercise 27 to prove that if T is diagonalizable, then so is $\overline{T}$.

29. Prove that if both T_W and $\overline{T}$ are diagonalizable and have no common eigenvalues, then T is diagonalizable.

30. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$, let $T = L_A$, and let W be the cyclic subspace of $\mathbb{R}^3$ generated by e_1.

    (a) Compute the characteristic polynomial of T_W.

    (b) Show that $\{e_2 + W\}$ is a basis for $\mathbb{R}^3/W$, and use this fact to compute the characteristic polynomial of $\overline{T}$.

    (c) Use the results of (a) and (b) to find the characteristic polynomial of A.

31. Recall that if T is a operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix, then the characteristic polynomial for T splits. Prove the converse of the above statement that if the characteristic polynomial of T splits, then there is an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

    Hints: Apply mathematical induction to $\dim(V)$. First prove that T has an eigenvector v. let $W = span(\{v\})$, and apply the induction hypothesis to $\overline{T} : V/W \to V/W$.

Exercises 32 through 39 are concerned with direct sums.

32. Let T be a linear operator on a vector space V, and let $W_1, W_2, \ldots, W_k$ be T-invariant subspaces of V. Prove that $W_1 + W_2 + \cdots + W_k$ is also a T-invariant subspace of V.

33. Let T be a linear operator on a finite-dimensional vector space V, and let $W_1, W_2, \ldots, W_k$ be $T-$invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. For each i, let β_i be an ordered basis for W_i, and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. Let $A = [T]_\beta$ and $B_i = [T_{W_i}]_{\beta_i}$, for $i = 1, 2, \ldots, k$. Then prove that $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$.

    Hint : Give a direct proof for the case $k = 2$ and extend it using mathematical induction on k, the number of subspaces.

34. Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.

35. Let T be a linear operator on a finite-dimensional vector space V, and let $W_1, W_2, \ldots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Prove that

$$\det(T) = \det(T_{W_1}) \det(T_{W_2}) \cdots \det(T_{W_k}).$$

36. Let T be a linear operator on a finite-dimensional vector space V, and let $W_1, W_2, \ldots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$. Prove that T is diagonalizable if and only if T_{W_i} is diagonalizable for all i.

37. Let $\mathcal{C}$ be a collection of diagonalizable linear operators on a finite dimensional vector space V. Prove that there is an ordered basis β such that $[T]_\beta$ is a diagonal matrix for all $T \in \mathcal{C}$ if and only if the operators of $\mathcal{C}$ commute under composition. (This is an extension of Exercise 24.)

    Hints for the case that the operators commute: The result is trivial if each operator has only one eigenvalue. Otherwise, establish the general result by mathematical induction on $\dim(V)$, using the fact that V is the direct sum of the eigenspaces of some operator in $\mathcal{C}$ that has more than one eigenvalue.

38. Let $B_1, B_2, \ldots, B_k$ be square matrices with entries in the same field, and let $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Prove that the characteristic polynomial of A is the product of the characteristic polynomials of the B_i's.

39. Let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}.$$

Find the characteristic polynomial of A.

Hint: First prove that A has rank 2 and that $span(\{(1,1,\ldots,1),(1,2,\ldots,n)\})$ is L_A-invariant.

40. Let $A \in M_{n\times n}(\mathbb{R})$ be the matrix defined by $A_{ij} = 1$ for all i and j. Find the characteristic polynomial of A.

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